Digital Communications
Equalization

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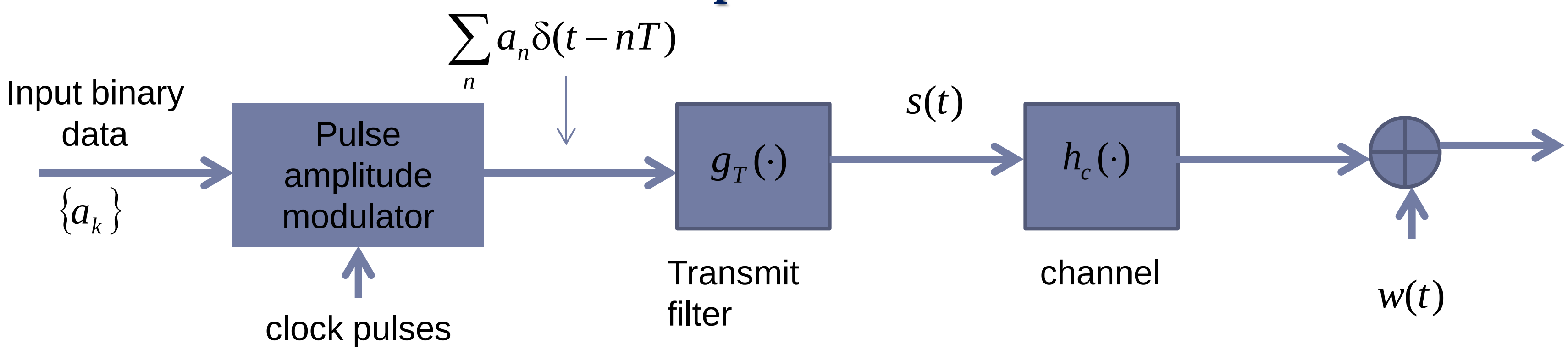
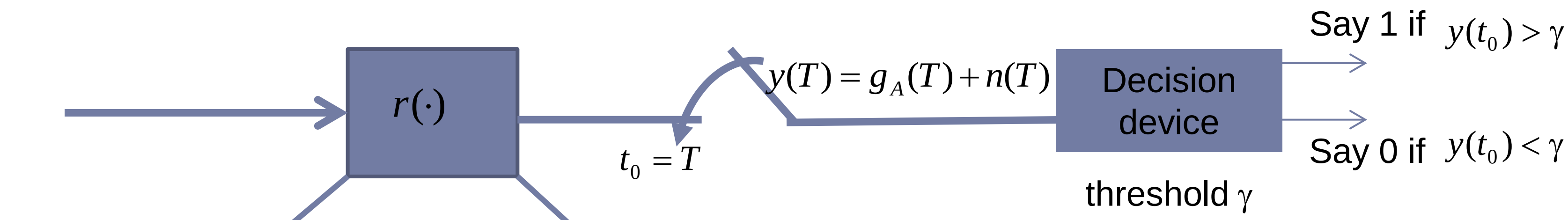
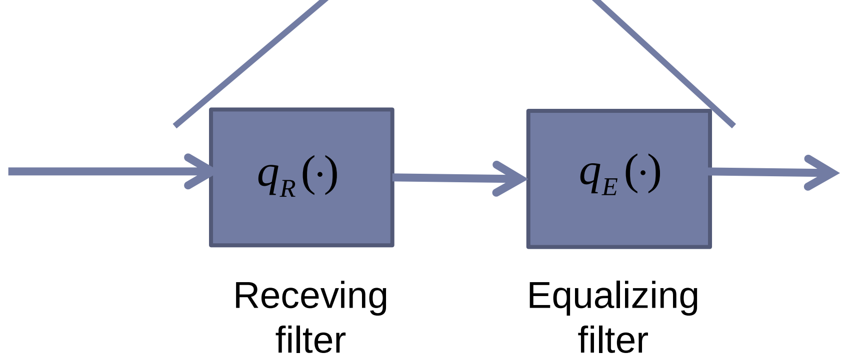
DIGITAL COMUNICATION SYSTEM
Optimal receiver







Receiving filter is essentially the demodulator, which recover the baseband pulse with the best possible signal-to-noise ratio.
Equalizing filter is an optional block needed in those systems where the channel-induced ISI can distort the signal

Baseband binary data transmission system

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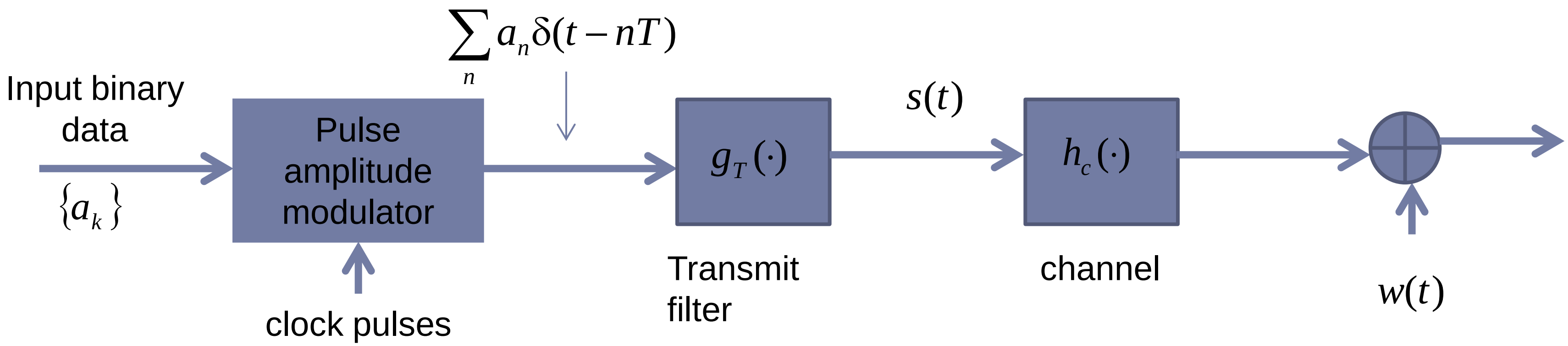
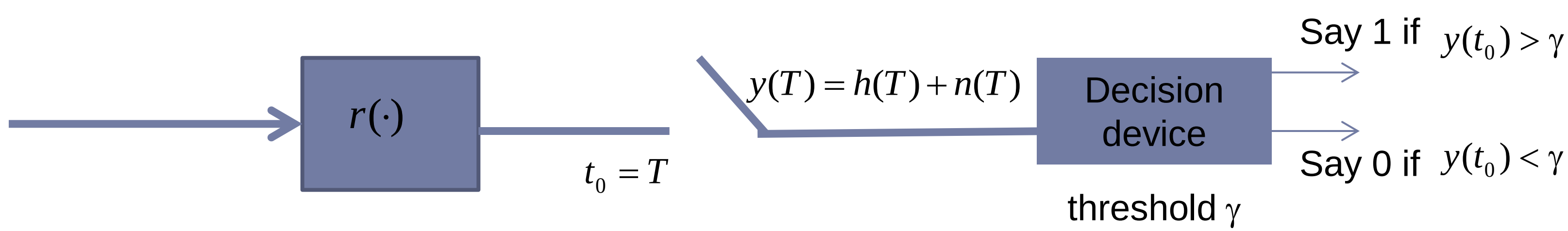
Optimum linear receiver in presence of noise and ISI





$\{a_k\}$ Stationary with autocorrelation $\alpha_m = E[a_n a_{n+m}]$

$w(t)$ Stationary with zero mean and power spectral density $|Q(f)|^2$

$\{a_k\}$ $w(t)$ Statistically independent

$h(t) = g_T(t) * h_c(t) * r(t)$ is assumed to be real (extension to the complex case can be done by the reader)

Optimum linear receiver in presence of noise and ISI

Given $G_T(f)$ and $H_c(f)$ (we are assuming that the channel is KNOWN at the receiver….and time-invariant!!!!!)

choose $R(f)$

that minimizes the Mean Square Error (MSE)

$$MSE = E\left[(y_k - a_k)^2\right]$$

Note: the real problem should be to minimize $P\left[\hat{a}_k \neq a_k\right]$
However, it is a very difficult problem and if $E\left[(y_k - a_k)^2\right] << 1$ *then* y_k *is very close to* a_k *in mean square sense and hence, in most of the cases it is is possible to assume that the*

$$\mu(y_k) = \hat{a}_k \equiv a_k$$

Optimum linear receiver in presence of noise and ISI

$$y_k = \sum_i a_i h((k-i)T) + w(kT) = \sum_i a_i h_{k-i} + w_k$$

$$MSE = E\left[(y_k - a_k)^2\right] = E\left[\left(\sum_i a_i h_{k-i} + w_k - a_k\right)^2\right]$$

$$= \sum_i \sum_j E\left[a_i a_j h_{k-i} h_{k-j}\right] + \sigma^2 + \alpha_0 + \cancel{2\sum_i E\left[a_i h_{k-i} w_k\right]} - 2\sum_i E\left[a_i h_{k-i} a_k\right] - \cancel{2\sum_i E\left[w_k a_k\right]}$$

$$MSE = \sigma^2 + \alpha_0 + \sum_i \sum_j \alpha_{i-j} h_{k-i} h_{k-j} - 2\sum_i \alpha_{k-i} h_{k-i}$$

$$p = k - i, q = k - j$$

$$MSE = \sigma^2 + \alpha_0 + \sum_p \sum_q \alpha_{p-q} h_p h_q - 2\sum_i \alpha_p h_q$$

Optimum linear receiver in presence of noise and ISI

Going in the frequency domain

$$M(f)=\sum_i \alpha_i e^{-j2\pi iT} \qquad U(f)=G_T(f)H_c(f)$$

being real

$$h_k=\int_{-\infty}^{\infty}H(f)e^{j2\pi fkT} \Rightarrow h_k=\int_{-\infty}^{\infty}H^*(f)e^{-j2\pi fkT}$$

$$\sigma^2=\int_{-\infty}^{\infty}|Q(f)|^2|R(f)|^2df=\int_{-\infty}^{\infty}|Q(f)|^2\frac{|H(f)|^2}{|U(f)|^2}df$$

$$\sum_p\sum_q \alpha_{p-q}h_ph_q=\sum_p\sum_q\alpha_{p-q}\int_{-\infty}^{\infty}H^*(f)e^{-j2\pi fpT}df\,h_q$$

$$=\int_{-\infty}^{\infty}H^*(f)\sum_q\left(\sum_p\alpha_{p-q}e^{-j2\pi fpT}\right)h_q df$$

$$=\int_{-\infty}^{\infty}H^*(f)\sum_q M(f)e^{-j2\pi fqT}h_q df$$

$$=\int_{-\infty}^{\infty}H^*(f)M(f)\frac{1}{T}\sum_l H\left(f-\frac{l}{T}\right)df$$

$$-2\sum_p\alpha_p h_p=-2\sum_p\alpha_p\int_{-\infty}^{\infty}H^*(f)e^{-j2\pi fpT}df$$

$$=-2\int_{-\infty}^{\infty}H^*(f)M(f)df$$

Optimum linear receiver in presence of noise and ISI

Let us summarize

$$MSE=\int_{-\infty}^{\infty}|Q(f)|^2\frac{|H(f)|^2}{|U(f)|^2}df+\alpha_0+\frac{1}{T}\int_{-\infty}^{\infty}M(f)H^*(f)\sum_l H\left(f-\frac{l}{T}\right)df$$

$$-2\int_{-\infty}^{\infty}M(f)H^*(f)df$$

Given $U(f)$

$$\min_{R(f)}\text{MSE}=\min_{H(f)=R(f)U(f)}\text{MSE}$$

$$con\quad \mathrm{H}_{\mathrm{opt}}(f):\quad \frac{\partial\text{MSE}}{\partial\text{H(f)}}=0$$

$$\text{siccome}\quad \frac{\partial\text{MSE}}{\partial\text{H(f)}}=2\frac{|Q(f)|^2}{|U(f)|^2}H(f)+\frac{2}{T}M(f)\sum_l H(f-\frac{l}{T})-2M(f)=0$$

Product of periodic terms of 1/T

periodic term of 1/T

Optimum linear receiver in presence of noise and ISI

$$G(f) \hat{=} \frac{|Q(f)|^2}{|U(f)|^2} H(f)$$

Must be a periodic function of 1/T

Condition of optimum:

$$G(f) + \frac{1}{T} M(f) \sum_l \underset{\substack{\| \\ G(f)}}{G\left(f - \frac{l}{T}\right)} \left| \frac{U\left(f - \frac{l}{T}\right)}{Q\left(f - \frac{l}{T}\right)} \right|^2 - M(f) = 0$$

Since it is periodic….therefore, I can bring it outside the sum

Optimum linear receiver in presence of noise and ISI

Solution

$$G(f)=\frac{M(f)}{1+M(f)\frac{1}{T}\sum_l \left|\frac{U\left(f-\frac{l}{T}\right)}{Q\left(f-\frac{l}{T}\right)}\right|^2}$$

so:

$$H_{opt}(f)=G(f)\frac{|H(f)|^2}{|Q(f)|^2}$$

Fundamental result

$$\boxed{R_{opt}(f)=\frac{H(f)}{U(f)}=\underbrace{\frac{U^*(f)}{|Q(f)|^2}}_{R_1(f)}\underbrace{\frac{M(f)}{1+M(f)\frac{1}{T}\sum_l \left|\frac{U\left(f-\frac{l}{T}\right)}{Q\left(f-\frac{l}{T}\right)}\right|^2}}_{R_2(e^{j2\pi fT})}}$$

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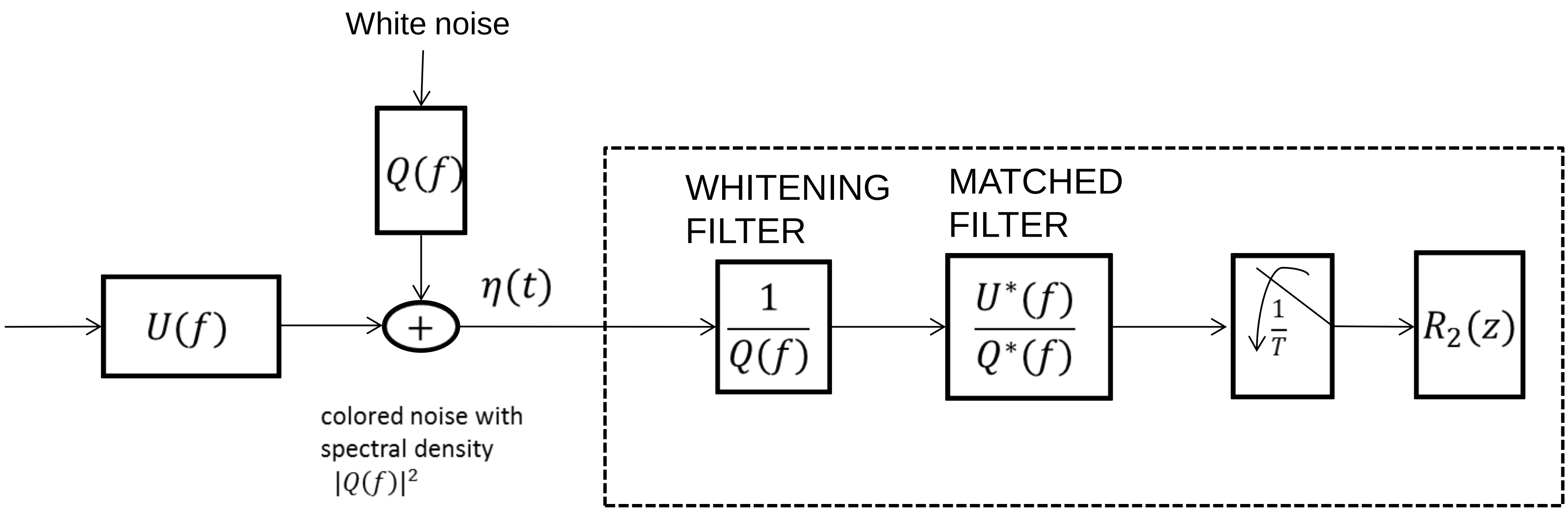
Optimum linear receiver in presence of noise and ISI



Optimum linear receiver in presence of noise and ISI

Hypothesis: white noise $\Rightarrow$ $|Q(f)|^2 = N_0$ constant

Let us define

$$\Phi\left(e^{j2\pi fT}\right) = \sum_p \varphi(pT) e^{-j2\pi fpT}$$

where

$$\varphi(t) = \int_{-\infty}^{\infty} u(x)u(x+t)dx$$ Autocorrelation of the pulse u(t)

$$U(f) = F[u(t)] \Rightarrow \Phi\left(e^{j2\pi fT}\right) = \frac{1}{T}\sum_l \left| U\left(f - \frac{l}{T}\right)\right|^2$$

$$\boxed{R_{opt}(f) = U^*(f)\frac{M(f)}{\underbrace{N_0 + M(f)\Phi\left(e^{j2\pi fT}\right)}_{R_2(e^{j2\pi fT})}}}$$

matched filter

Optimum linear receiver in presence of noise and ISI

Let us semplify $R_2(e^{j2\pi fT})$

It is a periodic function, so we can write the Fourier series:

$$R_2(e^{j2\pi fT}) = \sum_{n=-\infty}^{\infty} c_n e^{-j2\pi fnT}$$

$$c_n = \frac{1}{T}\int_{-\infty}^{\infty} R_2(e^{j2\pi fT}) e^{j2\pi fnT} df$$

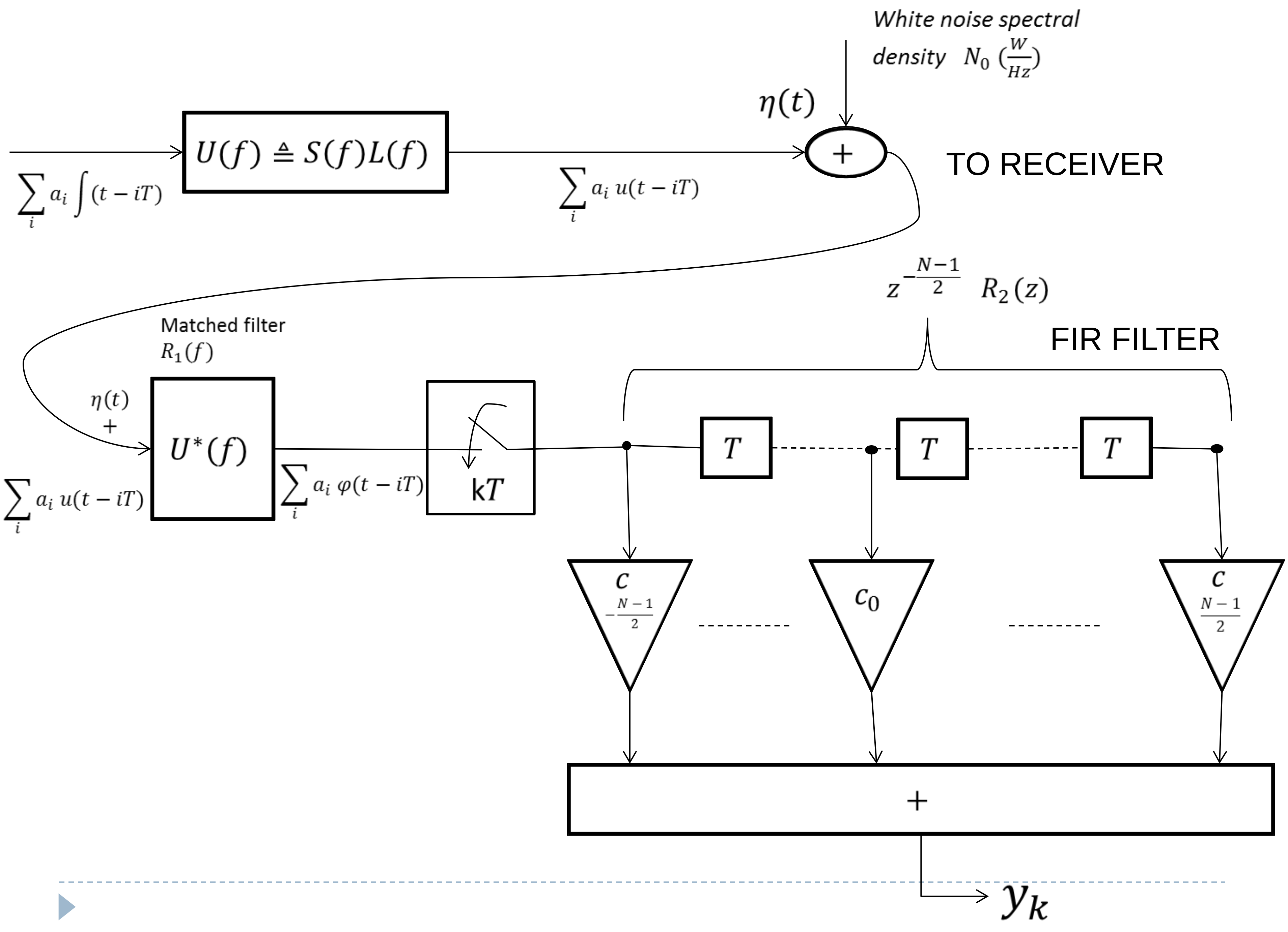
Let us truncate it up to the N most significant terms:

$$R_2(e^{j2\pi fT}) \approx \sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} c_n e^{-j2\pi fnT}$$

$$\Downarrow$$

$$R_2(z) \approx \sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} c_n z^{-n}$$

Transfer function of a FIR filter which introduce a delay of N-1/2



White noise spectral density $N_0\ (\frac{W}{Hz})$
$\eta(t)$
$\sum_i a_i \int (t-iT)$
$U(f) \triangleq S(f)L(f)$
$\sum_i a_i\, u(t-iT)$
+
TO RECEIVER
$z^{-\frac{N-1}{2}}\ R_2(z)$
FIR FILTER
Matched filter
$R_1(f)$
$\eta(t)$
+
$\sum_i a_i\, u(t-iT)$
$U^*(f)$
$\sum_i a_i\, \varphi(t-iT)$
kT
T
T
T
$c_{-\frac{N-1}{2}}$
c_0
$c_{\frac{N-1}{2}}$
+
y_k

Optimum linear receiver in presence of noise and ISI

Let us considetr two specific cases

1) Uncorrelated zero mean symbols $M(f) = \alpha_0$

$$\boxed{R_{opt}(f) = U^*(f)\frac{\alpha_0}{N_0 + \alpha_0 \Phi\left(e^{j2\pi fT}\right)}}$$

In low noise conditions, i.e. $\frac{\alpha_0}{N_0} >> 1$

$$R_2(e^{j2\pi fT}) \approx \frac{1}{\Phi\left(e^{j2\pi fT}\right)} \Rightarrow \underbrace{R_2(e^{j2\pi fT})\Phi\left(e^{j2\pi fT}\right)}_{\text{Transform of the fundamental pulse after sampling before R2}} = \cos t.$$

The effect of R_2 is to force the ISI to zero after the matched filter

R_2 is called **EQUALIZER**

- <u>On the other hand, for moderate noise, there is always some ISI if we use the MSE criterion</u>

Optimum linear receiver in presence of noise and ISI

Moreover, always in case 1), if $\varphi(t)$ satisfies the Nyquist criteria

$$\varphi(iT) = \varphi(0)\delta(iT) \Rightarrow \Phi\left(e^{j2\pi fT}\right) = \varphi(0)$$

$$\boxed{R_2(e^{j2\pi fT}) = \frac{\alpha_0}{N_0 + \alpha_0\varphi_0} = \cos t.}$$

No equalization is needed

Optimum linear receiver in presence of noise and ISI

2) Correlated zero mean symbols $M(f) \neq \alpha_0$

3) Nyquist criterion for the pulse $\varphi(iT) = \varphi(0)\delta(iT) \Rightarrow \Phi\left(e^{j2\pi fT}\right) = \varphi(0)$

The equalizer is needed

$$\boxed{R_2(e^{j2\pi fT}) = \frac{M(f)}{N_0 + \alpha_0 M(f)}}$$

The correlation introduces some ISI to compensate the noise

Correlation is used to reduce the effect of noise

Optimum linear receiver in presence of noise and ISI

2) Correlated zero mean symbols $M(f) \neq \alpha_0$

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Correlation is used to reduce the effect of noise

Alternative Derivation of the optimum MSE receiver in presence of noise and ISI

Hypothesis

1) White noise with power spectral density

2) $R_1(f) = U^*(f)$

3) $E\left[a_i a_j\right] = \alpha_0 \delta_{ij}, \quad E\left[a_i \eta_k\right] = 0 \quad \forall i, k$

The frequency response to the fundamental pulse in the decision point is:

$$\left|U(f)\right|^2 \xleftrightarrow{F^{-1}} \varphi(\mathrm{t})$$ (autocorrelation of the pulse *u(t)*)

The power spectral density of the noise just before sampling is:

$$N_0 \left|U(f)\right|^2$$

Alternative Derivation of the optimum MSE receiver in presence of noise and ISI

The autocorrelation of the sampled process $x_k = \sum_i a_i \varphi_{k-i} + n_k$

is given by:

$$R_x(m) = E\left[x_{k+m} x_k^*\right] = \underbrace{\alpha_0 \sum_n \varphi_{n+m} \varphi_n^*}_{\text{autocorrelation of the sampled PAM signal}} + \underbrace{N_0 \varphi_m}_{\text{autocorrelation of the sampled noise}}$$

The correlation between x_k and a_k

$$V(m) = E\left[a_{k+m} x_k^*\right] = \sum_i E\left[a_{k+m} a_i^*\right] \varphi_{k-i}^* + \cancel{E\left[a_{k+m} n_k^*\right]} = \alpha_0 \varphi_{-m}^*$$

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MSE algorithm

$$J = E\left[|e_k|^2\right] = E\left[|y_k - a_k|^2\right] = E\left[\left|\sum_{j=-\frac{N-1}{2}}^{\frac{N-1}{2}} c_j x_{k-j} - a_k\right|^2\right]$$

$$\boxed{\frac{\partial J}{\partial \underline{c}_l} = 0} \Rightarrow 2E\left[(y_k - a_k)x_{k-l}^*\right] = 0 \Rightarrow \sum_{j=-\frac{N-1}{2}}^{\frac{N-1}{2}} c_j \underbrace{E\left[x_{k-j}x_{k-l}^*\right]}_{R_x(l-j)} = \underbrace{E\left[a_k x_{k-l}^*\right]}_{V(l)}$$

$$\Downarrow$$

$$\sum_{j=-\frac{N-1}{2}}^{\frac{N-1}{2}} c_j \left[\alpha_0 R_\varphi(l-j) + N_0 \varphi_{l-j}\right] = \alpha_0 \varphi_{-l}^*$$

MSE algorithm

Let us consider the case of an ideal equalizer $N \to \infty$

Denoting with $\Phi(z) = Z[\varphi_n]$

The previous equation can be written in terms of zeta-transforms as follows:

$$C(z)\left[\alpha_0 \Phi(z)\Phi^*\left(\frac{1}{z}\right) + N_0\Phi(z)\right] = \alpha_0\Phi^*\left(\frac{1}{z}\right)$$

φ_k is an autocorrelation function $\Rightarrow$ is Hermitian $\varphi_k = \varphi^*_{-k}$

$$\Phi(z) = \Phi^*\left(\frac{1}{z^*}\right)$$

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MSE algorithm

$$C(z)\left[\alpha_0\Phi(z)+N_0\right]=\alpha_0$$

$$\Downarrow$$

$$C_{opt}(z)=\frac{\alpha_0}{\alpha_0\Phi(z)+N_0}$$